MATH 223.3 2006-2007 (01) (03) (05)
TEST #1 - Solutions
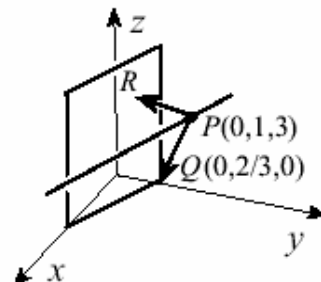**Integer scores *only*.**

PART A – 3 points each – total 30
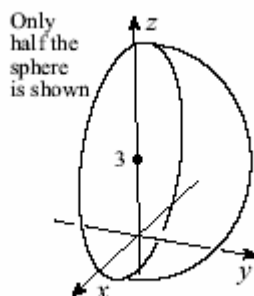
A1

The required distance d is the projection of
PQ on the direction **PR** normal to the plane:

$$d = \mathbf{PQ} \cdot \widehat{\mathbf{PR}}$$

$$= (0, -1/3, -3) \cdot \frac{(-3, -6, 0)}{\sqrt{9 + 36}}$$

$$= \frac{2}{3\sqrt{5}}.$$
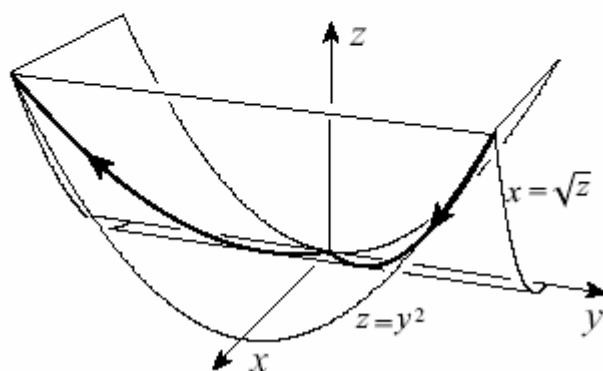

A2.   The sphere has centre (0, 0, 3) and radius 1.

A3.

If we set $y = t$, then parametric and
vector equations for the curve are
$x = |t|$, $y = t$, $z = t^2$, and
$\mathbf{r} = |t|\hat{\mathbf{i}} + t\hat{\mathbf{j}} + t^2\hat{\mathbf{k}}$.

A4.

From $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = (2\cos t, -2\sin t, 1)$, we obtain the unit tangent vector

$$\hat{\mathbf{T}} = \frac{(2\cos t, -2\sin t, 1)}{\sqrt{4\cos^2 t + 4\sin^2 t + 1}} = \frac{(2\cos t, -2\sin t, 1)}{\sqrt{5}}.$$

A5.

A vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{1}{\sqrt{5}}(-2\sin t, -2\cos t, 0) = -\frac{2}{\sqrt{5}}(\sin t, \cos t, 0),$$

and therefore $\hat{\mathbf{N}} = -(\sin t, \cos t, 0)$. The binormal is

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = -\frac{1}{\sqrt{5}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2\cos t & -2\sin t & 1 \\ \sin t & \cos t & 0 \end{vmatrix} = \frac{1}{\sqrt{5}}(\cos t, -\sin t, -2).$$

A6.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (1, 2t, 2t) \qquad |\mathbf{v}| = \sqrt{1 + 4t^2 + 4t^2} = \sqrt{1 + 8t^2} \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = (0, 2, 2)$$

The normal component of velocity is always zero, and the tangential component of velocity is speed $|\mathbf{v}| = \sqrt{1 + 8t^2}$.

$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{8t}{\sqrt{1 + 8t^2}} \qquad a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \left(8 - \frac{64t^2}{1 + 8t^2}\right)^{1/2} = \frac{2\sqrt{2}}{\sqrt{1 + 8t^2}}$$

A7.

If we set $z = x - y$, then $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\sin(x - y)}{x - y} = \lim_{z\to 0} \frac{\sin z}{z} = 1.$

The function is not continuous at $(1, 1)$ (being undefined there).

A8. $\dfrac{\partial f}{\partial x} = 3x^2 e^y - z\cos y \quad \Longrightarrow \quad \dfrac{\partial^2 f}{\partial z \partial x} = -\cos y$

or

$\dfrac{\partial f}{\partial z} = -x\cos y - \cos(y + z) \quad \Longrightarrow \quad \dfrac{\partial^2 f}{\partial x \partial z} = -\cos y$

as expected, $\dfrac{\partial^2 f}{\partial x \partial z} = \dfrac{\partial^2 f}{\partial z \partial x}$

A9.

If $\nabla f = (2xy - y)\hat{\mathbf{i}} + (x^2 - x)\hat{\mathbf{j}}$, then $\dfrac{\partial f}{\partial x} = 2xy - y$ and $\dfrac{\partial f}{\partial y} = x^2 - x$. From the first equation, we can say that $f(x, y) = x^2 y - xy + \phi(y)$, where $\phi(y)$ is any differentiable function of y. To determine $\phi(y)$ we substitute this expression for $f(x, y)$ into the second equation, $x^2 - x + d\phi/dy = x^2 - x$. Consequently, $d\phi/dy = 0$, and therefore $\phi(y) = C$, a constant. Thus, $f(x, y) = x^2 y - xy + C$.

A10.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$
$$= (2x - 3x^2 y^2)\cos\theta + (-2yx^3)\sin\theta$$